Image and Vision Computing 29 (2011) 167–177

Contents lists available at ScienceDirect

Image and Vision Computing

journal homepage: www.elsevier.com/locate/imavis

Radial basis function based level set interpolation and evolution for deformable modelling

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ARTICLE INFO

Article history:
Received 2 November 2009
Received in revised form 25 May 2010
Accepted 18 August 2010

Keywords:
Deformable model
Level set
Radial basis function
Re-initialisation free

ABSTRACT

We present a study in level set representation and evolution using radial basis functions (RBFs) for active contour and active surface models. It builds on recent works by others who introduced RBFs into level sets for structural topology optimisation. Here, we introduce the concept into deformable models and present a new level set formulation able to handle more complex topological changes, in particular perturbation *away from the evolving front*. In the conventional level set technique, the initial active contour/surface is implicitly represented by a signed distance function and periodically re-initialised to maintain numerical stability. We interpolate the initial distance function using RBFs on a much coarser grid, which provides great potential in modelling in high dimensional space. Its deformation is considered as an updating of the RBF interpolants, an ordinary differential equation (ODE) problem, instead of a partial differential equation (PDE) problem, and hence it becomes much easier to solve. Re-initialisation is found no longer necessary, in contrast to conventional finite difference method (FDM) based level set approaches. The proposed level set updating scheme is efficient and does not suffer from self-flattening while evolving, hence it avoids large numerical errors. Further, more complex topological changes are readily achievable and the initial contour or surface can be placed arbitrarily in the image. These properties are extensively demonstrated on both synthetic and real 2D and 3D data. We also present a novel active contour model, implemented with this level set scheme, based on multiscale learning and fusion of image primitives from vector-valued data, e.g. colour images, without channel separation or decomposition.

1. Introduction

Ever since Kass et al. [1] introduced the active contour or snake model, there has been a multitude of works on the development of active contour models, some theoretical and some tuned to certain applications. Traditional snakes suffer from several issues, such as limited capture range and difficulties in reaching concavities. The application of the level set method [2] to the active contour model has enabled the latter to adapt to complex topologies. It avoids the need to reparameterise the curve and the contours are able to split or merge in order to capture an unknown number of objects without resorting to dedicated contour tracking. However, the original level set based active contour [3] has proved to be of limited use in real applications as it assumes that contours reach the object boundaries at roughly the same time, and thus, it often suffers from weak edge leakage.

A great deal of research has been carried out to innovate and improve external forces in order to overcome its shortcomings. Xu and Prince's work [4], a boundary based approach, greatly improved the snake model on convergence issues. They iteratively diffused gradient vectors from edge areas to homogeneous regions to increase the capture range and to improve its performance towards concavities. This gradient vector flow (GVF) snake model is less sensitive to noise interference and has better ability in recovering weak edges. Convergence issues, however, still exist particularly when the evolving contours are tangent to the gradient vectors as noted by several authors, e.g. [5,6]. Over the last few years, region based methods, such as [7–11], have become very popular as they are generally less initialisation dependent and exhibit better ability in handling textures and image noise interference. For example, in [8], Chan and Vese considered the active contour as an energy minimisation of a Mumford–Shah based minimal partition problem. Some practical applications can be found in [7,12], amongst many others.

The extension of the active contour model into the active surface model is relatively straightforward when using implicit representation based on the level set scheme, e.g. [13]. However, this implicit representation embeds the contour or surface into a higher dimensional space which needs to be updated iteratively as a whole, becoming more computationally expensive than traditional parametric approaches. The evolution of the embedded contour or surface is solved using partial differential equations (PDEs) which in most

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doi:10.1016/j.imavis.2010.08.011

cases involve costly finite difference methods (FDM). It also requires more memory storage as usually the level set grid is collocating with the image grid and needs to update each grid point instead of only the points on the deforming contour or surface.

More importantly, in conventional level set methods (i.e. those based on upwind FDM [2], e.g. the classic geodesic model [3]), active contours or surfaces are not able to create topological changes *away from the zero level set* where the deformable contours or surfaces are embedded [12,14]. This means, for example, that the level sets would miss holes inside objects. In order to accurately solve the associated PDEs using FDM, a local method, the implicit function is required to be smooth and remain so while evolving. Thus, re-initialisation is usually necessary in order to achieve numerical stability. Although alternative methods without re-initialisation are available, such as [15,16], they often require dedicated extension of the speed function defined on the contour.

As a primary interpolation tool, radial basis functions (RBFs) have received increasing attention in solving PDE systems in recent years. For example, Cecil et al. [17] used RBFs to generalise conventional FDM on a non-uniform (unstructured) computational grid to solve the high dimensional Hamilton–Jacobi PDEs with high accuracy. Very recently, Wang et al. [14] interpolated level set functions using RBFs and transformed the Hamilton–Jacobi PDEs into a system of ordinary differential equations (ODEs) for structural topology optimisation in 2D.

In this paper, we adapt the approach presented for structure design in [14] to apply to active modelling and show how our proposed model greatly enhances the performance of active models. Following [14], we interpolate the initial level set function using RBFs and treat the implicit contour/surface propagation as an ODE problem, which is much easier and more efficient to solve. However, the updating scheme proposed in [14] is unsuitable for active modelling and in this paper we propose a simple, but effective, normalisation scheme to resolve this issue. This new active model exhibits significant improvements in initialisation invariancy, convergence, and topology adaptability. The initial contour or surface is embedded into an implicit function derived from the distance transform in the way same as the conventional level set approach. However, we then interpolate it using RBFs which can be placed on a much coarser grid, and with the interpolation characterised by its expansion coefficients. Thus, deforming the original implicit function is achieved by updating the expansion coefficients. Re-initialisation during evolution is no longer necessary and perturbation away from the zero level is possible to obtain more sophisticated topological changes. This has a significant impact on the active contour and surface models to free them from initialisation restrictions. *The contour or surface can therefore be initialised anywhere in the image*. Furthermore, we demonstrate that the segmentation can be carried out even without any initial contours.

We show an implementation of this RBF level set method in a new region based active contour model. Its external force field is derived through statistical modelling of colour pixels in multiscale and treats the colour image in full 3D, without channel separation or decomposition, in order to simultaneously capture spatial and spectral interactions. It also has the potential to model multi-band data with more than three channels.

Additionally, we show that it is convenient for the proposed RBF based level set method to extend from the 2D image domain to 3D space, i.e. extension from active contours to active surfaces.

Recently in [18], Morse et al. placed RBFs at contour landmarks to implicitly represent the active contour, thereby avoiding the manipulation of a higher dimensional function. However, their method requires dynamic insertion and deletion of landmarks which is non-trivial. Similar to the parametric representation, the resolution and position of the landmarks can affect the accuracy of contour representation. More recently, the authors in [19] presented a similar idea to ours which we preliminarily presented in [20]. Following [14], [19] used direct RBF interpolation and evolution without speed normalisation, which will create artificial contours during evolution. Comparison of level set evolution will show our proposed method provides more stable evolution with considerably fewer numerical errors.

The remainder of this paper is organised as follows. In the next section we present a brief review of the conventional level set method, RBF interpolation, the proposed RBF level set evolution, its application to a region based active contour model on colour images, and its extension to 3D. Experimental results on both synthetic and real world data are presented in Section 3. Conclusions and future work are discussed in Section 4.

2. Proposed method

2.1. Level set representation

In level set representation [2], a deformable contour or surface is implicitly represented by a multi-dimensional scalar function (signed distance field) with the moving front embedded at the zero level set. Let C and Φ denote the moving front and the level set function respectively. The relationship between these two can be expressed as:

$$C = \{\mathbf{x} | \Phi(\mathbf{x}) = 0 \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, and subject to $\Phi(\mathbf{x}) > 0$ for $\mathbf{x}$ inside the front and $\Phi(\mathbf{x}) < 0$ for $\mathbf{x}$ outside. This representation is parameter free and intrinsic. Considering the front (contour or surface) evolving according to $dC/dt = F\mathcal{N}$ for a given function F (where $\mathcal{N}$ denotes the inward unit normal), then the embedding function should deform according to $\partial\Phi/\partial t = F|\nabla\Phi|$, where F is computed on the level sets. By embedding the evolution of C in that of Φ, topological changes of C, such as splitting and merging, are handled automatically.

The level set function is commonly initialised using the signed distance transform and its evolution numerically solved using FDM with the upwind scheme [2]. The numerical errors using this local approximation method may gradually accumulate and can contaminate the solution. Thus, periodic re-initialisation of the level set function is usually applied to maintain numerical stability. The conventional level set method generally prevents topological changes taking place away from the developing front which restricts other forms of topological changes, such as developing holes inside objects. The method presented in this paper avoids solving the PDE problem and transforms it into a much simpler ODE problem. It will allow the level set contour or surface to deal with regions away from the evolving front by initiating new fronts in the level set and thus capture holes or inner boundaries of objects. This makes the active contour or surface framework not only much more successful but also initialisation invariant.

2.2. RBF interpolated level set function

Radial basis functions have attracted significant attention in scattered data interpolation in multi-dimensions, as well as in other applications such as data classification. For example, Mullan et al. [21] used dipole RBF representation with RBF centres slightly above and below the implicit surface to propagate the 3D surface. Turk and O'Brien [22] introduced constraint points to obtain implicit surfaces using RBFs, which were applied to implicit active contour modelling by Morse et al. [18].

Similar to recent works by Cecil et al. [17] and Wang et al. [14], we interpolate the level set function $\Phi(\mathbf{x})$ using a certain number of RBFs. Each RBF, ψ_i, is a radial symmetric function centred at position $\mathbf{x}_i$. Only a single function ψ is used to form this family of RBFs. The multiquadric spline, found to be one of the best for RBF interpolation [23] is used here, with the RBFs then written as follows:

$$\psi_i(\mathbf{x}) = \psi(|\mathbf{x}-\mathbf{x}_i|) = \sqrt{|\mathbf{x}-\mathbf{x}_i|^2 + c_i^2}, \quad (2)$$

where c_i is usually treated as a constant for all RBFs. The interpolation of the level set function is expressed as follows:

$$\Phi(\mathbf{x}) = p(\mathbf{x}) + \sum_{i=1}^{N} \alpha_i \psi_i(\mathbf{x}), \quad (3)$$

where N denotes the number of RBFs, α_i are the expansion coefficients of the corresponding RBF, and $p(\mathbf{x})$ is a first-degree polynomial, which in the 2D case can be written as $p(\mathbf{x}) = p_0 + p_1 x + p_2 y$.[1] To ensure a unique solution to this RBF interpolation, the expansion coefficients must satisfy as follows:

$$\sum_{i=1}^{N} \alpha_i = \sum_{i=1}^{N} \alpha_i x_i = \sum_{i=1}^{N} \alpha_i y_i = 0. \quad (4)$$

These N number of RBFs are distributed across the domain and their centre values, denoted by $f_1,...,f_N$, are given by the level set function. The RBF interpolant then can be obtained by solving the following linear system:

$$\mathbf{H}\alpha = \mathbf{f}, \quad (5)$$

where

$$\mathbf{H} = \begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix}, \quad \alpha = [\alpha_1 \cdots \alpha_N p_0 p_1 p_2]^T, \quad \mathbf{f} = [f_1 \cdots f_N \quad 0 \quad 0 \quad 0]^T, \quad (6)$$

and

$$\mathbf{A}_{i,j} = \psi_j(\mathbf{x}_i), \quad i,j = 1,\dots,N, \qquad \mathbf{P}_{i,j} = p_j(\mathbf{x}_i), \quad i = 1,\dots,N, j = 1,2,3, \quad (7)$$

where p_j are the basis for the polynomial. Thus, the RBF interpolation of the level set function in (3) can be written as follows:

$$\Phi(\mathbf{x}) = \Psi^T(\mathbf{x})\alpha, \quad (8)$$

where $\Psi(\mathbf{x}) = [\psi_1(\mathbf{x}) \cdots \psi_N(\mathbf{x}) 1 x y]^T$ and the expansion coefficients α is given by: $\alpha = \mathbf{H}^{-1}\mathbf{f}$.

2.3. Active modelling using RBF level set

As stated in Section 2.1, the deformation of the active contour is achieved by propagating the level sets along their normal directions according to a localised speed which is usually image dependent. It can be expressed as the following PDE:

$$\frac{\partial \Phi}{\partial t} + F|\nabla \Phi| = 0, \quad (9)$$

where F is the speed function along the normal direction. Unlike the conventional level set method, here we have a level set function interpolated by RBFs. Following [14], we assume that time and space are separable and the time dependence of the level set function is now due to the RBF interpolation, i.e. the expansion coefficients. Updating the level set function is now considered as updating the RBF expansion coefficients. In other words, the expansion coefficients become time dependent:

$$\Phi = \mathbf{\Psi}^T(\mathbf{x})\alpha(t). \quad (10)$$

Thus, the level set updating function (9) can be re-written as follows:

$$\mathbf{\Psi}^T \frac{d\alpha}{dt} + F|(\nabla \mathbf{\Psi})^T \alpha| = 0. \quad (11)$$

This indicates that the original PDE problem can now be treated as an ODE problem. The spatial derivative $\nabla \mathbf{\Psi}$ can be solved analytically. Given (5), (11) can be re-written as follows:

$$\mathbf{H}\frac{d\alpha}{dt} + \mathbf{B}(\alpha) = 0, \quad (12)$$

where

$$\mathbf{B}(\alpha) = \begin{bmatrix} F(\mathbf{x}_1)\left|\left(\nabla \mathbf{\Psi}^T(\mathbf{x}_1)\right)\alpha\right| \\ \vdots \\ F(\mathbf{x}_N)\left|\left(\nabla \mathbf{\Psi}^T(\mathbf{x}_N)\right)\alpha\right| \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (13)$$

The solution can be obtained by iteratively updating the expansion coefficients using the first order Euler's method, also adopted from [14]:

$$\alpha\left(t^{n+1}\right) = \alpha(t^n) - \Delta t \mathbf{H}^{-1}\mathbf{B}(\alpha(t^n)). \quad (14)$$

The updating of the level set function starts from interpolating its initial state using RBFs. As usual, the initial level set function is obtained from the signed distance transform and is often initialised such that $|\nabla \Phi| = 1$. Then RBFs are spread across the domain and the interpolation takes place, providing us with the initial value of the expansion coefficients, α. The interpolated level set is then evolved according to (14) and (3). In conventional level set methods, the level set function is initialised and maintained as a signed distance function through re-initialisation. The re-initialisation can prevent numerical errors, accumulated through level set updating using the FDM in solving the associated PDE, from corrupting the solution. Moreover, the signed distance function is not a solution to the level set formulation, e.g. the classic geodesic formulation, which means the level set function will not remain a signed distance function in the process of contour evolution and hence periodic re-shaping of the level set is necessary [24]. In a variational approach such as that proposed by Chan and Vese [8], the level set function hardly remains a signed distance function at any time of the evolution. We similarly do not impose such a constraint. In fact, in order to develop perturbations away from the existing front, it is necessary for those level sets that are far away from the zero level to deviate from a signed distance function [25]. In our approach, the level set function is interpolated using RBFs and its shape is determined by RBF expansion coefficients. The evolution of the level set function is formulated to be an ODE problem as shown earlier, with level set derivatives solved analytically, instead of locally approximated using FDM as in conventional level set methods. This gives stability in level set evolution and allows continuous updating without the need for re-initialisation, and the level set updating becomes much simpler and more efficient.

Although (14) has been proven useful in structure optimisation in [14], a direct application of this updating scheme was found to be unsuitable for active contour models. The updating of the level set function is achieved by adjusting the RBF interpolation expansion coefficients as shown in (12)–(14), with the change in coefficients

[1] For simplicity, we present the solution in 2D. Its solution in higher dimensions is straightforward.

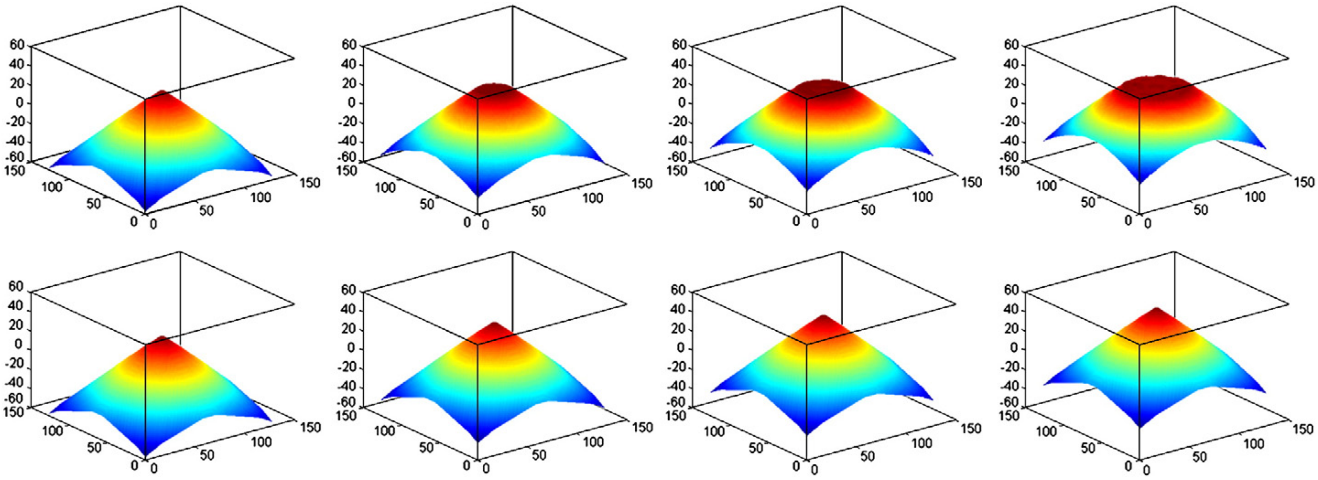

Fig. 1. Updating RBF level set using non-normalised and normalised schemes — first row: non-normalised scheme; second row: proposed normalised scheme.

directly related to the spatial derivative of the level set, which is computed analytically. It is very likely that there exist stationary points in the level set function with their spatial derivative magnitude close to zero. This results in drastically slow changes of level set values in those regions, which gradually spread to their neighbours, and hence leads to a flattening effect. Level set evolution can thus be hampered and it can lead to further updating issues when the level set function becomes more complex. An example is given in Fig. 1 where a circular shape is embedded in an initial level set function, a cone shape as shown in the first image in the first row. A constant force is applied to this active contour, i.e. F is a real constant. This force expands the contour outwards which should generally lift the level set function. However, as shown in the first row, the top of the level set function becomes stationary and gradually turns into a largely flat surface. This is due to the gradient magnitude of the RBF interpolated level set at those points being close to zero ($|(\nabla\Psi^T(\mathbf{x}_i))\alpha|\rightarrow 0$) and based on (13) and (14) the expansion coefficients at those places would evolve much slower. As a result, the level set function tends to get flattened and this is particularly undesirable when topological changes should be taking place — see for example in Fig. 2 where two circles are expanding due to the same constant expanding force. The valley in the level set function is affected and introduces substantial numerical artifacts, and finally contaminates the solution as shown in the last image in the first row, indicated by the highly irregular spikes in the level set function. Special care is thus necessary, for example using dedicated velocity extension.

Fortunately, in deformable modelling the direction of the speed along the normal has dominant effect on the final segmentation, not its magnitude (however, it is preferable for level set evolution that the speed is smoothly varying in the spatial domain). Since the gradient of the level set function is generally smoothly varying, we propose a simple yet highly effective solution to solve this problem which is to modify the speed function by "normalising" it against the local gradient estimated from the RBF interpolants, i.e. the following:

$$F^{'}(\mathbf{x}_i) = \frac{1}{|(\nabla\boldsymbol{\Psi}^T(\mathbf{x}_i))\alpha|}F(\mathbf{x}_i). \qquad (15)$$

Note that due to this global modelling using RBFs, the gradient is dependent on all the RBF centres across the domain, instead of local neighbours. Thus, the gradient near the advancing front is unlikely to be zero, i.e. this normalisation will be unlikely to disturb the developing front. In effect, we use level set spatial derivative magnitude values as inverse weightings for the speed function in

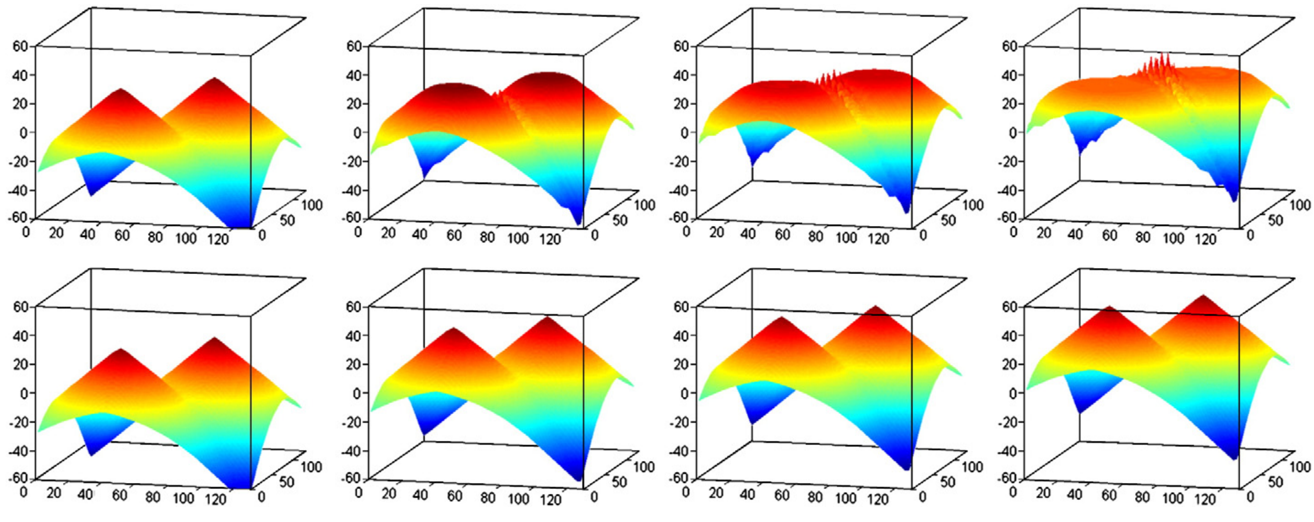

Fig. 2. Updating the RBF level set using non-normalised and normalised schemes — first row: non-normalised scheme; second row: proposed normalised scheme. See Fig. 3 for corresponding contour evolution.

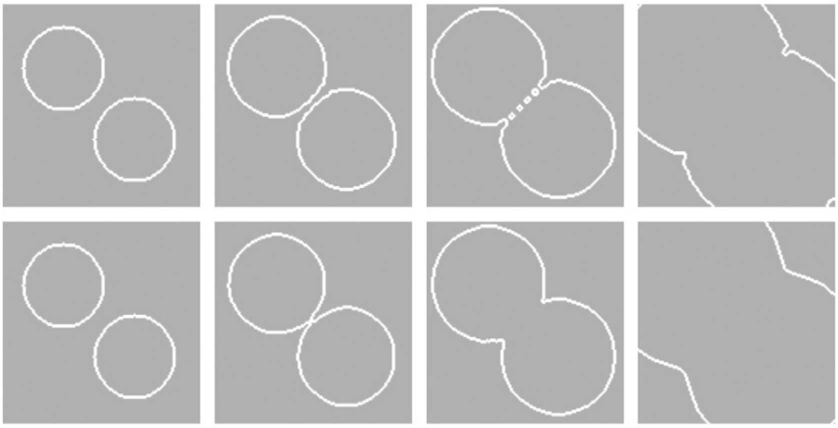

Fig. 3. Contour evolution using non-normalised and normalised schemes — first row: non-normalised scheme; second row: proposed normalised scheme. See Fig. 2 for corresponding level set evolution.

coefficient updating. Thus, if the spatial derivative magnitude is close to zero the external force will be scaled up in order to have a larger influence in coefficient changes; hence it prevents the level set from flattening. For regions where the magnitude of the level set gradient is close to one, the speed function is hardly changed. Using this normalisation scheme, (13) then simplifies to the following:

$$\mathbf{B}(\alpha) = [F(\mathbf{x}_1) \ \dots \ F(\mathbf{x}_N) \ 0 \ 0 \ 0]^T. \tag{16}$$

Updating the expansion coefficients and hence the level set are now even simpler and more efficient. The second row in Fig. 1 shows the results using the normalised approach. The level set function does not get flattened while updating the expansion coefficients. Topological changes, for example merging, shown in the second row in Fig. 2, can be conveniently handled, in contrast to the non-normalised scheme (shown in the first row). Fig. 3 shows the corresponding contour evolution. It can be clearly seen that the non-normalised version produces erroneous contours and irregularities, while the proposed approach handles topological changes correctly.

One of the main advantages of using an RBF interpolated level set to represent an active contour is that more sophisticated topological changes, besides merging and splitting, can be readily achieved. Let $\mathcal{F}$ be a region indication function, i.e. $\mathcal{F}<0$ for points inside an object of interest and $\mathcal{F}>0$ for the rest. In Fig. 4, the object of interest is shown in dark grey, and the initial snake is drawn in white. The snake using the conventional level set scheme with re-initialisation fails to recover the hole in the object as periodic re-initialisation prevents it from doing so. The proposed RBF based level set method successfully recovers the shape without dedicated effort in monitoring the front propagation. This occurs because the proposed method uses RBF interpolants to estimate the level set gradient, a global estimation instead of a local one. Front propagation is then unlikely to introduce oscillation around the zero level set. Thus re-initialisation is not necessary to maintain stability. The proposed RBF expansion coefficients updating scheme prevents other level sets, away from the evolving front, from flattening themselves so that these level sets are sensitive enough to sufficient gradient changes for the RBF interpolated front to grow new fronts (i.e. contours or surfaces). It is possible that shocks occur in certain force evolutions which may result in steep level set surfaces. However, our empirical results show that the proposed scheme is generally robust due to its intrinsic smoothness from RBF interpolation.

Notably, variational level set methods, such as the Chan–Vese model [8], can also capture internal boundaries. However, the use of the delta function in these methods, even with a smoothed (regularised) version, means that it has a very local support in the vicinity of the zero level set, which inevitably leads to irregularities in level set evolution as those levels close to zero evolve much faster than the rest. This will also hamper the level set evolution as more levels are pushed away from the zero level [25].

2.4. A region based active contour model using RBF level sets

We now present a novel region based active contour model using the proposed RBF level set method. As mentioned earlier in Section 1, region based methods generally perform better in the presence of weak edges and image noise interference. More importantly in relevance to this work, region based methods are considered much less initialisation dependent. Thus, when compared against conventional level set evolution, there is no factor other than the level set scheme affecting its evolution and convergence. There are two classes of popular region based approaches. One is based on the well-known

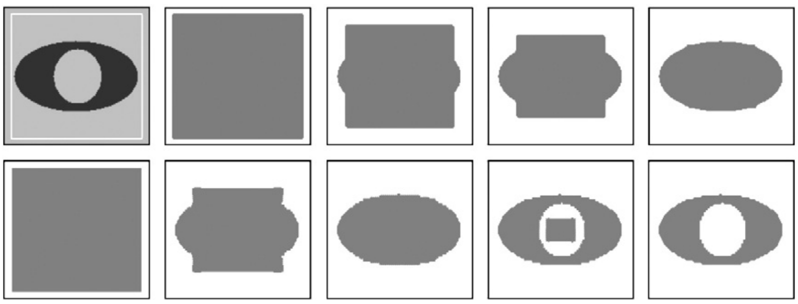

Fig. 4. More complex topological changes are readily achievable — first row: initial snake and recovered shape using conventional level set method; second row: recovered shape using proposed method. The final images in both rows show the stabilised results.

Mumford–Shah formulation [26], where the contours compete with each other while preserving the piecewise smooth assumption. The other, such as the works in [7,9], globally model the image data and the active contour evolves to maximise its posterior. We opt for the second approach and model the image data using texem based mixture modelling [27,28]. However, our emphasis remains on demonstrating the performance of our proposed RBF level set method and give a comparative study with the conventional level set approach *in the same active contour framework*.

The external force of the proposed active contour model is based on our recently introduced texem model [27], which is suitable for vector-valued data, such as colour images. It handles the colour image in full 3D, without having to decompose it into separate channels, for example, using colour separation or principal component analysis. Thus, it takes into account both spatial interactions and spectral or inter-channel interactions simultaneously. Next, we briefly review the texem representation, discuss its multiscale learning and texem grouping, and present the active contour formulation.

2.4.1. Modelling colour images

Here, we give a very brief overview of the texem representation (see [27] for a detailed exposition). The texem (or texture exemplar) model is based on the assumption that a given image can be generated from a collection of image patches and the variation in placement results in appearance variations in the images. The texems are entities of those representative patches along with the statistics characterising their variations. Each of the texems learnt contains partial degrees of image micro-structures, i.e., they are implicit representations of image primitives.

Formally, each texem $\mathbf{m}$ is defined by a mean, μ, and a corresponding variance, ω, i.e. $\mathbf{m}=\{\mu,\omega\}$. An image is then considered as a superposition of patches of various sizes. To learn the texems, the original image $\mathbf{I}$ is broken down into a set of P overlapping patches $\mathbf{Z}=\{\mathbf{Z}_i\}_{i=1}^{P}$, each containing pixels from a subset of image coordinates. We assume that there exist K texems, $\mathcal{M}=\{\mathbf{m}_k\}_{k=1}^{K}$, $K\ll P$, for image $\mathbf{I}$ such that each patch in $\mathbf{Z}$ can be generated from a texem with certain added variations as follows:

$$p(\mathbf{Z}_i|\theta_k)=p(\mathbf{Z}_i|\mu_k,\omega_k)=\prod_{j\in S}\mathcal{N}\left(\mathbf{Z}_{j,i};\mu_{j,k},\omega_{j,k}\right), \tag{17}$$

where θ_k denotes the kth texem's parameters with mean μ_k and variance ω_k, $\mathcal{N}(.)$ is a Gaussian distribution over $\mathbf{Z}_{j,i}$, S is the patch pixel grid, $\mu_{j,k}$ and $\omega_{j,k}$ denote mean and variance at the jth pixel position in the kth texem. Using mixture modelling, we assume the following probabilistic model:

$$p(\mathbf{Z}_i|\Theta)=\sum_{k=1}^{K}p(\mathbf{Z}_i|\theta_k)\beta_k, \tag{18}$$

where $\Theta=(\beta_1,...,\beta_K,\theta_1,...,\theta_K)$, and β_k is the *priori* probability of kth texem constrained by $\sum_{k=1}^{K}\beta_k=1$. The Expectation Maximisation (EM) technique can be used to estimate the model parameters.

2.4.2. Multiscale learning and grouping

Clearly, each image patch from an image has a measurable relationship with each texem according to the *posteriori*, $p(\mathbf{m}_k|\mathbf{Z}_i,\Theta)$, which can be conveniently obtained using Bayes' rule:

$$p(\mathbf{m}_k|\mathbf{Z}_i,\Theta)=\frac{p(\mathbf{Z}_i|\mathbf{m}_k,\Theta)\beta_k}{\sum_{k=1}^{K}p(\mathbf{Z}_i|\mathbf{m}_k,\Theta)\beta_k}. \tag{19}$$

Thus, every texem can be viewed as an individual textural class component, and the *posteriori* can be regarded as the component likelihood. Choosing various sizes of neighbourhood ensures capturing image structures at various scales. Alternatively, we can build a pyramidical representation of the image and collecting colour pixels across scales for learning texems, which can be more efficient in incorporating neighbourhood relationships. A simple Gaussian pyramid was found to be sufficient. Thus, each pixel in the finest level can trace its parent pixel back to the coarsest level forming a unique route or branch. Eq. (17) in the texem representation can then be rewritten as:

$$p(\mathbf{Z}_i|\theta_k)=p(\mathbf{Z}_i|\mu_k,\omega_k)=\prod_{n\in l}\mathcal{N}\left(\mathbf{Z}_i^{(n)};\mu_k^{(n)},\omega_k^{(n)}\right), \tag{20}$$

where $\mathbf{Z}_i$ here is a branch of pixels, l is the number of levels in the multiscale pyramid, and $\mathbf{Z}_i^{(n)}$, $\mu_k^{(n)}$, and $\omega_k^{(n)}$ are the colour pixel at level n in ith branch, mean at level n of kth texem, and variance at level n of kth texem, respectively. This is essentially the same form as (17). However, the image is not partitioned into patches, but rather laid out in multiscale first and then separated into branches. The pixels are collected across scales, instead of from its neighbours.

Considering a textural region may contain multiple visual elements and display complex patterns, a single texem might not be able to fully represent such textural regions. Hence, several texems can be grouped together to jointly represent "multimodal" texture regions. Here, we use a simple but effective mixture model order reduction method proposed by Manduchi [29,30] to describe multimodal regions. Our basic strategy is to group some of the texems based on their spatial coherence. The grouped texem representation takes the following form:

$$\hat{p}(\mathbf{Z}_i|c)=\frac{1}{\hat{\beta}_c}\sum_{k\in G_c}p(\mathbf{Z}_i|\mathbf{m}_k)\beta_k,\quad \hat{\beta}_c=\sum_{k\in G_c}\beta_k, \tag{21}$$

where G_c is the group of texems that are combined together to form a new cluster c which labels the different texture classes, and $\hat{\beta}_c$ is the *priori* for new cluster c. The mixture model can thus be re-formulated as follows:

$$p(\mathbf{Z}_i|\Theta)=\sum_{c=1}^{\hat{K}}\hat{p}(\mathbf{Z}_i|\mathbf{m}_k)\hat{\beta}_c, \tag{22}$$

where $\hat{K}$ is the adjusted number of classes.

The grouping in (22) is carried out based on the assumption that the *posteriori* probabilities of some texems are typically spatially correlated and hence can be grouped together. The process should minimise the decrease of model descriptiveness, D, which is defined as [29,30]:

$$D=\sum_{j=1}^{K}D_j, \tag{23}$$

$$D_j=\int p\left(\mathbf{Z}_i|\mathbf{m}_j\right)p\left(\mathbf{m}_j|\mathbf{Z}_i\right)d\mathbf{Z}_i=\frac{E\left[p\left(\mathbf{m}_j|\mathbf{Z}_i\right)^2\right]}{\beta_j}, \tag{24}$$

where $E[.]$ is the expectation computed with respect to $p(\mathbf{Z}_i)$. In other words, the compacted model should retain as much descriptiveness as possible. This is known as the Maximum Description Criterion (MDC). The descriptiveness decreases drastically when well separated texem components are grouped together, but decreases very slowly when spatially correlated texem component distributions merge together. Thus, the texem grouping should search for smallest change in descriptiveness, ΔD. It can be carried out by greedily grouping two texem components, $\mathbf{m}_a$ and $\mathbf{m}_b$, at a time with minimum ΔD_{ab}:

$$\Delta D_{ab}=\frac{\beta_b D_a+\beta_a D_b}{\beta_a+\beta_b}-\frac{2E[p(\mathbf{m}_a|\mathbf{Z}_i)p(\mathbf{m}_b|\mathbf{Z}_i)]}{\beta_a+\beta_b}. \tag{25}$$

We can see that the first term in (25) is the maximum possible descriptiveness loss when grouping two texems, while the second term is the normalised cross correlation between the two texem component distributions. Since one texture region may contain different texem components that are significantly different to each other, it is beneficial to smoothen the *posteriori* as proposed in [30] such that a pixel that originally has high probability to belong to just one texem component will be softly assigned to a number of components that belong to the same “multimodal” region.

2.4.3. Active contour formulation

Once the multiscale texem learning and grouping is finished, the *posteriori* probability of the class of interest can be computed once again using Bayes' rule:

$$\hat{p}(c|\mathbf{Z}_i) = \frac{\hat{p}(\mathbf{Z}_i|c)\hat{\beta}_c}{\sum_{c=1}^{\hat{K}}\hat{p}(\mathbf{Z}_i|c)\hat{\beta}_c}, \tag{26}$$

where

$$\hat{p}(\mathbf{Z}_i|c)\hat{\beta}_c = \sum_{k\in G_c} p(\mathbf{Z}_i|\mathbf{m}_k)\beta_k. \tag{27}$$

To simplify the notation, let u denote the posterior probability of the class of interest. The posterior probability of region of interest can then be used as the external force for the active contour. For example, if the posterior probability is higher than the average expectation, the contour should expand in that region; otherwise, the contour should shrink itself. The colour texem based active contour can thus be formulated as follows:

$$\frac{dC}{dt} = w\kappa\mathcal{N} + \left(u - \frac{1}{m}\right)\mathcal{N}, \tag{28}$$

where w is a real constant, κ denotes the curvature, m is the number of classes and $\frac{1}{m}$ is the average expectation of a class probability. Its level set representation takes the following form:

$$\frac{\partial\Phi}{\partial t} = w\kappa|\nabla\Phi| + \left(u - \frac{1}{m}\right)|\nabla\Phi|. \tag{29}$$

We can then apply the proposed RBF level set scheme to solve (29). The curvature can be computed analytically. However, since the RBF interpolation is intrinsically smoothing the level set function, and also in the interest of examining the evolution under the external force field, we ignore this curvature based internal contour regularisation term, and use the image dependent force term alone to deform the active contour. The contour is supposed to expand and shrink to maximise the posterior of the regions of interest. Thus, (29) can be reformulated as (cf. (11)):

$$\mathbf{\Psi}^T\frac{d\alpha}{dt} + \left(\frac{1}{m} - u\right)|(\nabla\mathbf{\Psi})^T\alpha| = 0. \tag{30}$$

Its solution is obtained by solving (14), where $\mathbf{B}(\alpha) = \left[\left(\frac{1}{m} - u(\mathbf{x}_1)\right)\dots\left(\frac{1}{m} - u(\mathbf{x}_N)\right)0\ 0\ 0\right]^T$. Fig. 5 shows a flow chart of the proposed method, from texem learning to level set updating.

The RBF centres can be placed on a regular computational grid, which can be a lot coarser than the pixel grid due to the interpolation capability of the RBFs, unlike in the conventional level set approach where a dense computational grid is necessary in order to estimate the derivatives with sufficient accuracy. In the interest of further reducing the computational complexity, an irregular grid can be used and the grid structure can be determined using a multiscale approach, that is, start from a coarse level set interpolation using a coarse regular grid which gives a good indication of the topology and shape of the object of interest and then more RBFs centre are placed at and near object boundaries resulting in a much finer irregular grid that is compact and data driven.

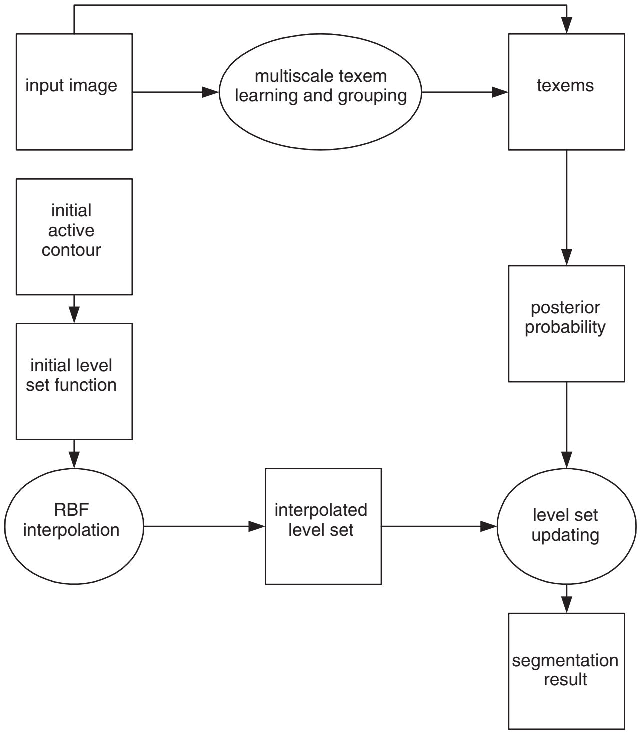

Fig. 5. Flow chart of the proposed active contour segmentation.

With this proposed level set updating scheme, new contours can even grow out in regions away from existing contours, which is not possible for the conventional level set approach. Equally importantly, the initial contour can be forced to vanish from the image domain while newly appearing fronts are able to localise the regions. This gives significant improvement in initialisation invariancy and achieves global minimum, instead of local minimum (as demonstrated earlier in Fig. 4).

2.5. Extension to 3D

Similar to the conventional level set method, the extension of the proposed method to higher dimensions is straightforward. Even better, the proposed method demands only a much coarser mesh grid. The RBF centres can be more loosely placed in 3D, instead of the full pixel grid often used in conventional level set approaches. Also, solving the ODE system in 3D is much easier than solving the PDE system. The updating of the expansion coefficients is efficient and again does not require re-initialisation of the level set function. The main computation cost comes from interpolating the initial level set and reconstructing the level set function after it stabilises. However, there are several methods available to speed up the process, such as the Fast Multipole Method (FMM) [31]. It has been shown by several authors, for example [32], that RBFs can be efficiently used in interpolating scattered points.

3. Experimental results

We illustrate the results of the proposed method on both synthetic and real data.[2] For most of the examples, it is assumed it is a

[2] Example animations are accompanied with this submission.

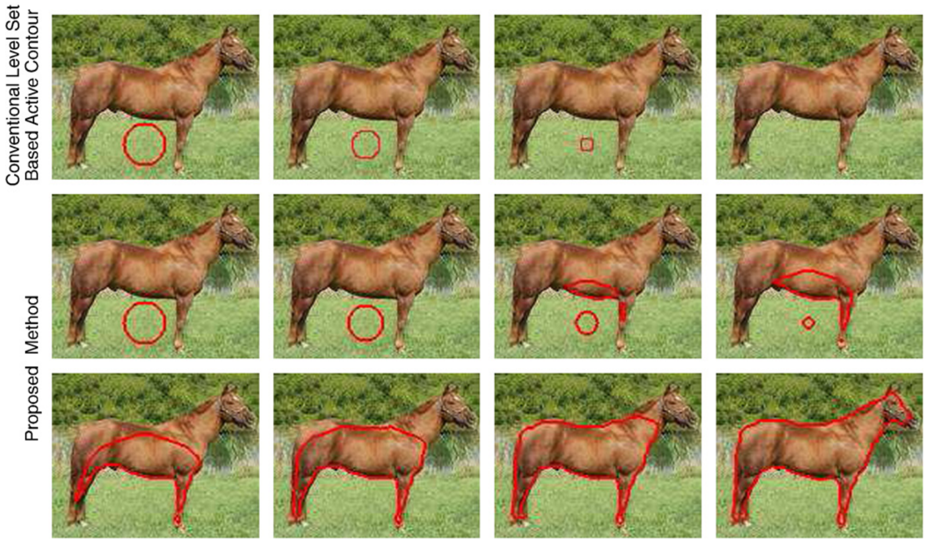

Fig. 6. Comparative result on real image — first row: segmentation result using conventional level set with the initial snake forced to shrink; second and third rows: results using proposed RBF level set method.

foreground and background separation problem as typical in deformable modelling. In the case of multi-textural regions, we do require prior knowledge of the region of interest when performing active contour based segmentation.

The proposed method was implemented using Matlab on a 2.5 GHz CPU PC with 8 GBs of RAM. A regular computational grid was used with one RBF centre in every 3×3 region of pixels. For the range of images we tested with, from 128×128 pixels to 300×200 pixels, it takes around 90 s to interpolate the level set function, around 150 s to learn the texems and perform texem grouping, and less than 10 s to update the level set function. This latter step is very efficient since it only needs to solve the ODE problem and the spatial derivatives are analytically computed. It is many times faster than conventional level set methods.

To recapitulate, in Fig. 4, the active contour in the conventional level set updating scheme could only recover the outer object boundary where the continuous front propagation could possibly reach. The proposed method, however, allowed the level set to create new components away from the existing propagating front and capture the inner boundary as well.

Fig. 6 shows the comparative results of the texem based snake using the conventional level set approach (top row) and the proposed RBF level set method (rows 2 and 3). The initial snake was placed outside the object of interest and was forced to *shrink*. The conventional method failed to localise the object while the proposed method succeeded by growing new contours inside the object. In this case, the conventional method requires the initial snake to be specifically overlapping or placed inside the object. This assumes that prior knowledge of the spatial position of the object of interest, as well as its topology, is available for initialisation, which is not always the case in real world applications. The proposed method does not require any such assumptions and is particularly useful in detecting unknown number of objects with complex topologies.

Another comparative example is given in Fig. 7, where multiple regions exist with more complex topologies, i.e. internal boundaries. The proposed method localised all the regions that were indicated by

Fig. 7. Comparative result on real image — first row: segmentation result using conventional level set; second and third rows: results using proposed RBF level set method.

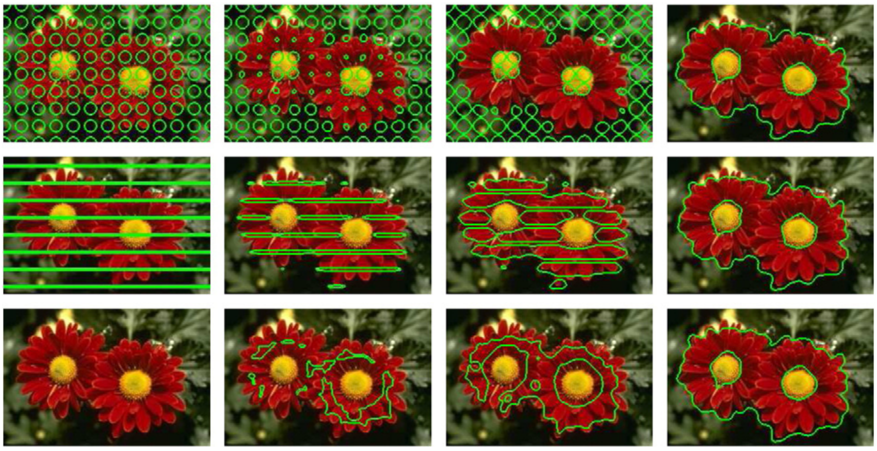

Fig. 8. Initialisation invariancy — first row: segmentation using the proposed method with multiple uniformly distributed circular initial active contours; second row: segmentation using the proposed method with initial active contours as parallel lines; third row: segmentation using the proposed method without any initial active contour, i.e. completely initialisation-free.

the external force field, while the conventional level sets completely failed with such an initialisation.

Although logically straight forward, it is significant to realise that the ability to develop new contours in fact indicates there is actually no need to place the initial contour in the first place. The final row of Fig. 8 gives such an example. The first and second rows of Fig. 8 also provide two completely different initialisations with the first one commonly used in region based methods as an approach to initialisation invariancy.

More experimental results of using the proposed method are provided in Fig. 9. In each case, the proposed level set updating scheme facilitated the active contour to capture all object boundaries of interest. Also note that the texem based region force handled regional colour and feature variations very well. Fig. 10 provides several more initialisation-free examples, where the proposed method carried out the segmentation without placing an initial contour, successfully developing new contours and localising the objects. Again note that the texem based region force handled regional colour and feature variations very well.

Next, the ability of the proposed method in handling complex 3D topologies and initialisation invariancy is examined. We applied the 3D RBF level set method on synthetic data and evolved the active surface according to (9), where $\mathcal{F}$ is the region indication function, i.e. $\mathcal{F}<0$ for regions inside the 3D objects and $\mathcal{F}>0$ otherwise, as before. In Fig. 11, the target object was a hollow sphere. The initial surface was placed to surround the object and was forced to shrink to capture the

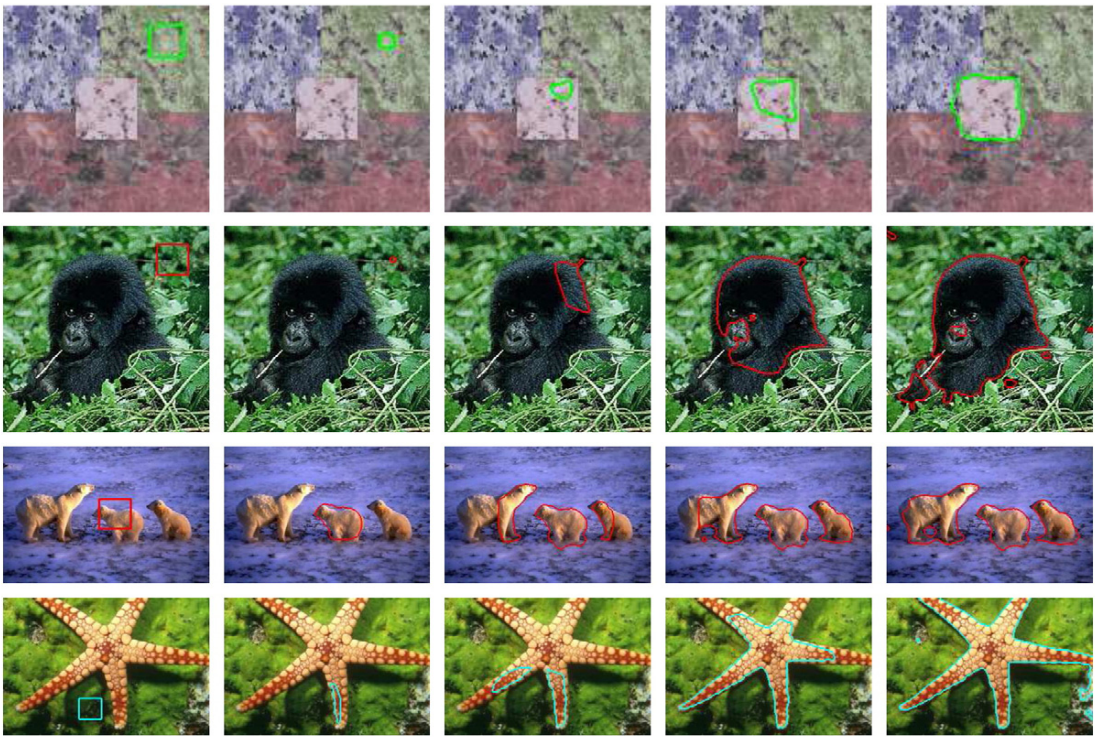

Fig. 9. Segmentation results using the proposed method.

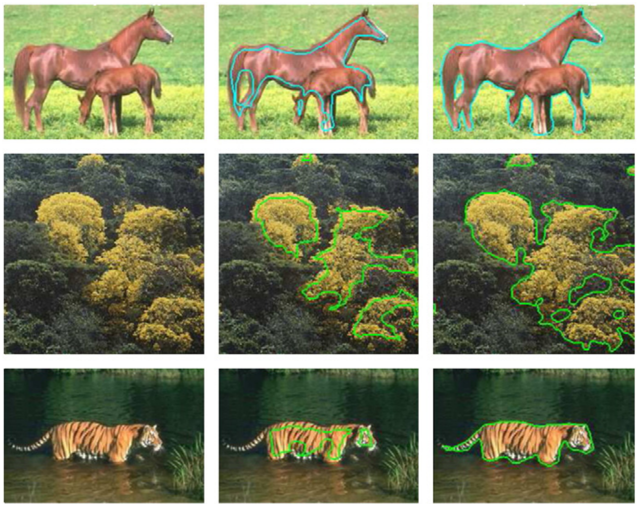

Fig. 10. More examples of segmentation without initialisation using the proposed method.

object boundaries. With the proposed RBF level set method, not only was the outer boundary localised, but also the boundary inside was captured, i.e. as the active surface was deforming, a new zero level set developed inside the object. The next example given in Fig. 12 shows that the region indication function shrinks the active surface that initialised outside the target object. There was no intersection between the initial surface and the object, or when the initial surface deformed and disappeared. With the conventional level set approaches, the surface would shrink and disappear completely, as it does in the first phase of the proposed method as shown in the first three images in the upper row. However, the proposed method allows the level set to deform further to "grow" outside the initial surface and finally recovers the object. This again demonstrates the method's initialisation independence feature. In the final example shown in Fig. 13, we demonstrate the ability of the proposed method in modelling very complex geometry in 3D.

4. Conclusion

We have presented a novel method to perform implicit modelling using RBFs. The proposed method has a number of advantages over the conventional level set scheme: (a) The evolution of the level set function is considered as an ODE problem rather than a much more difficult PDE problem; (b) Re-initialisation of the level set function was found no longer necessary for this application; (c) More complex topological changes, such as holes within objects, are comfortably found; (d) The active contour and surface models using this technique are initialisation independent; (e) The computational grid can be much coarser, hence it is potentially much more computationally efficient when updating the level set function, particularly in high

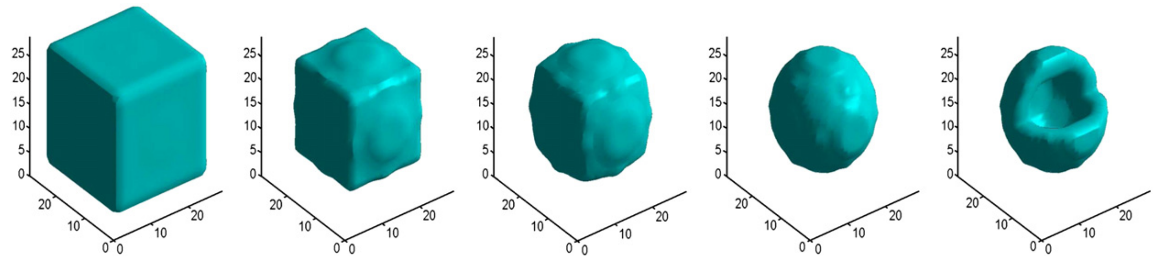

Fig. 11. Recovering a hollow sphere using a proposed method — from left: initial deformable surface, evolving deformable surface, stabilised surface, and the stabilised surface with a section cut away to show the hole captured inside.

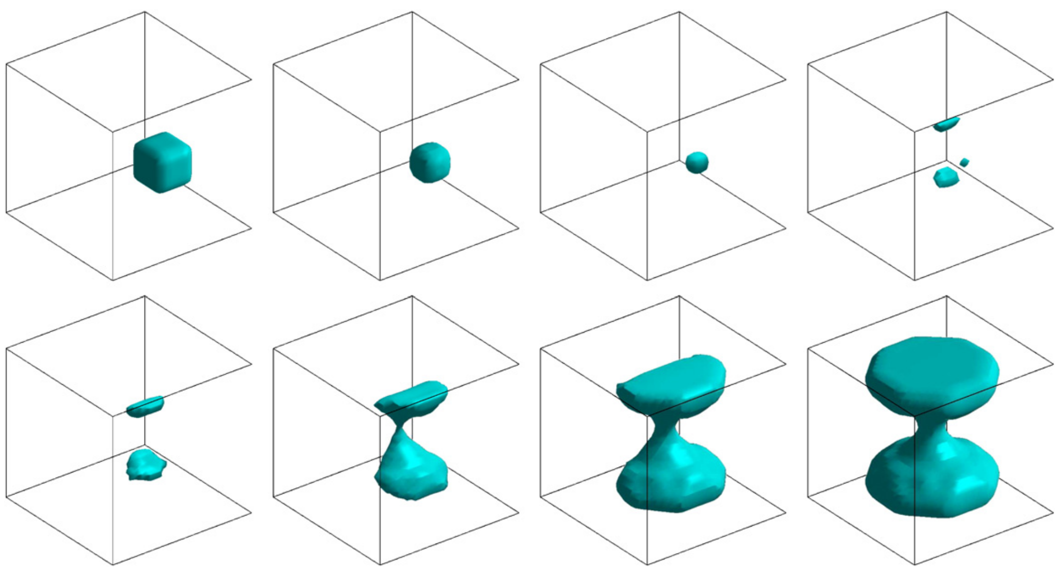

Fig. 12. Arbitrary initialisation — the initial surface is placed outside the object and is forced to shrink, but the proposed method allows the level set to deform further to develop a zero level set outside the initial surface and recover the object.

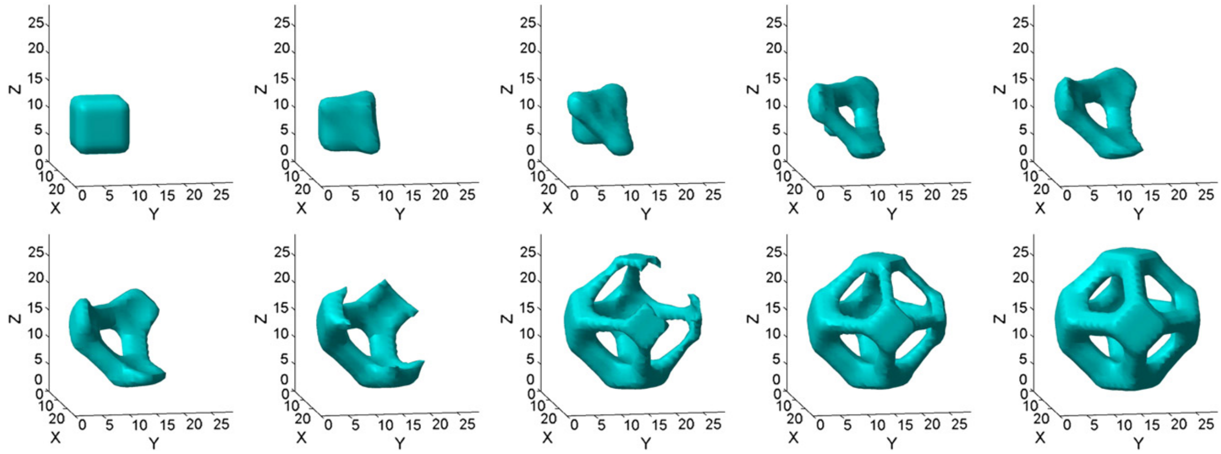

Fig. 13. Recovering a complex 3D shape.

dimensional spaces. The proposed texem based active contour method models the colour image without decomposing it into a lower dimensional space and shows promising performance on segmenting textured colour images.

Appendix A. Supplementary data

Supplementary data to this article can be found online at doi:10.1016/j.imavis.2010.08.011.

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